Tutorial 5

1. Let C denote the positively oriented unit circle $|z| = 1$ about the origin. Show that if $f(z)$ is the principal branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\operatorname{Log} z\right] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of $z^{-3/4}$ and $g(z)$ is the following branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\log z\right] \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the same power function, then

$$\int_C f(z)\ dz \neq \int_C g(z)\ dz.$$

Solution. We write $z(\theta) = e^{i\theta}$ and it is clear to have

$$f(z(\theta))z'(\theta) = ie^{i\theta/4} = \sin\frac{\theta}{4} + i\cos\frac{\theta}{4},$$

which is piecewise continuous [1] on either branch so the integrals exist.

By direct calculations, we have

$$\int_C f(z)\ dz = i\int_{-\pi}^{\pi} e^{i\theta/4}\ d\theta = 4\sqrt{2}i$$

and

$$\int_C g(z)\ dz = i\int_0^{2\pi} e^{i\theta/4}\ d\theta = 4i - 4.$$

2. Show that if C is the boundary of the triangle with vertices at the points $0, 3i$, and -4, oriented in the counterclockwise direction, then the following integral

$$\int_C (e^z - \overline{z})\ dz$$

is bounded.

Solution.

We first observe that

$$|e^z - \overline{z}| \leq |e^z| + |\overline{z}| \leq e^x + \sqrt{x^2 + y^2}.$$

Since $e^x \leq 1$ for $x \leq 0$ and the distance $\sqrt{x^2 + y^2}$ from the origin is always less than or equal to 4. Thus,

$$|e^z - \overline{z}| \leq 5$$

when z is on C. The length of C is evidently 12. Hence

$$\left|\int_C (e^z - \overline{z})\ dz\right| \leq 5 \cdot 12 = 60.$$

[1] a piecewise continuous function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits.

3. Let C_R denote the upper half of the circle $|z| = R$ ($R > 2$), taken in the counterclockwise direction. Find the value of the following limit

$$\lim_{R\to\infty} \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz.$$

Solution. If $|z| = R$ with $R > 2$, then

$$|2z^2 - 1| \leq 2|z|^2 + 1 = 2R^2 + 1$$

and

$$|z^4 + 5z^2 + 4| = |z^2 + 1| \cdot |z^2 + 4| \geq \Big|\, |z|^2 - 1 \,\Big| \cdot \Big|\, |z|^2 - 4 \,\Big| = (R^2 - 1)(R^2 - 4).$$

Hence,

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \leq \pi R \cdot \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)} \to 0$$

as $R \to \infty$.